Numerical simulations [356] performed in the framework of homogeneous turbulence have shown a persistence of anisotropy at the small scales. However, it should be noted that this anisotropy is detected only on statistical moments of the velocity field of order three or more, with first- and second-order moments being isotropic.

5.3.2 Anisotropic Models

Here we describe the main models proposed in the anisotropic framework. Except for Aupoix's spectral model, none of these take explicit account of the backward cascade mechanism. They are:

1. Aupoix's spectral model (p.153), which is based on the anisotropic EDQNM analysis. The interaction terms are evaluated by adopting a preset shape of the energy spectra and subgrid mode anisotropy. This model, which requires a great deal of computation, has the advantage of including all the coupling mechanisms between large and small scales.
2. Horiuti's model (p.154), which is based on an evaluation of the anisotropy tensor of the subgrid modes from the equivalent tensor constructed from the highest frequencies in the resolved field. This tensor is then used to modulate the subgrid viscosity empirically in each direction of space. This is equivalent to considering several characteristic velocity scales for representing the subgrid modes. This model can only modulate the subgrid dissipation differently for each velocity component and each direction of space, but does not include the more complex anisotropic transfer mechanisms through the cutoff.
3. The model of Carati and Cabot (p.155), who propose a general form of the subgrid viscosity in the form of a fourth-rank tensor. The components of this tensor are determined on the basis of symmetry relations. However, this model is a applicable only when the flow statistically exhibits an axial symmetry, which restricts its field of validity.
4. The model of Abba *et al.* (p.156) which, as in the previous example, considers the subgrid viscosity in the form of a fourth-rank tensor. The model is based on the choice of a local adapted reference system for representing the subgrid modes, and which is chosen empirically when the flow possesses no obvious symmetries.
5. Models based on the idea of separating the field into an isotropic part and inhomogeneous part (p.157), in order to be able to isolate the contribution of the mean field in the computation of the subgrid viscosity, for models based on the large scales, and thereby better localize the information contained in these models by frequency. This technique, however, is applicable only to flows exhibiting at least one direction of homogeneity.

Aupoix Spectral Model. In order to take the anisotropy of the subgrid scales into account, Aupoix [7] proposes adopting preset shapes of the energy spectra and anisotropy so that the relations stemming from the previously described EDQNM analysis of anisotropy can be used. Aupoix proposes the following model for the energy spectrum:

$$E(k) = K_0 \varepsilon^{2/3} k^{-5/3} \exp\left\{ f(k/k_d) \right\} \quad , \tag{5.50}$$

where

$$f(x) = \exp\left[-3.5x^2 \left(1 - \exp\left\{ 6x + 1,2 - \sqrt{196x^2 - 33.6x + 1.4532} \right\} \right) \right] \quad . \tag{5.51}$$

This spectrum is illustrated in Fig. 5.2. The anisotropy spectrum is modeled by:

$$\begin{aligned} H_{ij}(k) &= b_{ij} \left[5 + \frac{k}{E(k)} \frac{\partial E(k)}{\partial k} \right] \\ &\times \left[1 + \mathcal{H}\left(\frac{k}{k_{\max}} - 1 \right) \mathcal{H}\left(|\mathcal{F}(\overline{\mathbf{u}})| \right) \left\{ \left(\frac{k}{k_{\max}} \right)^{-2/3} - 1 \right\} \right] , \end{aligned} \tag{5.52}$$

where $\mathcal{F}(\overline{\mathbf{u}}) = \nabla \times \overline{\mathbf{u}}$, $k_{\max}$ is the wave number corresponding to the energy spectrum maximum, and $\mathcal{H}$ the Heaviside function defined by:

$$\mathcal{H}(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{otherwise} \end{cases} \quad ,$$

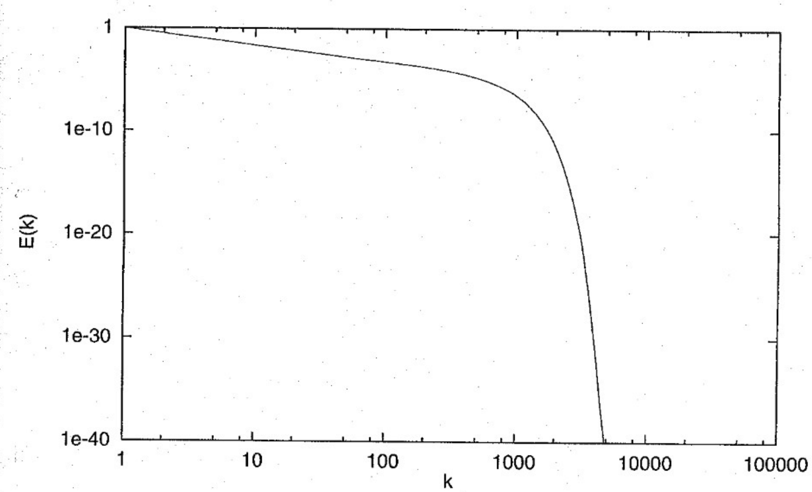

Fig. 5.2. Aupoix spectrum ($k_d = 1000$).

$$\nu^{(2)}_{ijkl} = \nu^{(2)}_{jikl} \quad ,$$
$$\nu^{(2)}_{ijkl} = -\nu^{(2)}_{jilk} \quad ,$$
$$\nu^{(2)}_{iikl} = 0 \quad . \tag{5.62}$$

The tensor $\nu^{(2)}$ therefore contains 15 independent coefficients, which raises the number of coefficients to be determined to 40.

Further reductions can be made using the symmetry properties of the flow. For the case of symmetry about the axis defined by the vector $\mathbf{n} = (n_1, n_2, n_3)$, the authors show that the model takes a reduced form that now uses only four coefficients, $C_1, ..., C_4$:

$$\begin{aligned} \tau^{\rm d}_{ij} &= -2C_1\overline{S}_{ij} - 2C_2\left(n_i\overline{s}_j + \overline{s}_i n_j - \frac{2}{3}\overline{s}_k n_k \delta_{ij}\right) \\ &- C_3\left(n_i n_j - \frac{1}{3}n^2\delta_{ij}\right)\overline{s}_k n_k - 2C_4\left(\overline{r}_i n_j + n_i \overline{r}_j\right) \quad , \end{aligned} \tag{5.63}$$

where $\overline{s}_i = \overline{S}_{ik} n_k$ and $\overline{r}_i = \overline{\Omega}_{ik} n_k$.

Adopting the additional hypothesis that the tensors $\nu^{(1)}$ and $\nu^{(2)}$ verify the Onsager symmetry relations for the covariant vector $\mathbf{n}$ and the contravariant vector $\mathbf{p}$:

$$\begin{aligned} \nu^{(1)}_{ijkl}(\mathbf{n}) &= \nu^{(1)}_{klij}(\mathbf{n}) \quad , \\ \nu^{(2)}_{ijkl}(\mathbf{n}) &= \nu^{(2)}_{klij}(\mathbf{n}) \quad , \\ \nu^{(1)}_{ijkl}(\mathbf{p}) &= \nu^{(1)}_{klij}(-\mathbf{p}) \quad , \\ \nu^{(2)}_{ijkl}(\mathbf{p}) &= \nu^{(2)}_{klij}(-\mathbf{p}) \quad , \end{aligned} \tag{5.64}$$

we get the following reduced form:

$$\tau^{\rm d}_{ij} = -2\nu_1\overline{S}^{\parallel}_{ij} - 2\nu_2 n^2 \overline{S}^{\perp}_{ij} \quad , \tag{5.65}$$

where ν_1 and ν_2 are two scalar viscosities and

$$\overline{S}^{\parallel}_{ij} = \frac{1}{n^2}\left(n_i\overline{s}_j + \overline{s}_i n_j\right) - \frac{1}{3n^2}\overline{s}_k n_k \delta_{ij}, \quad \overline{S}^{\perp}_{ij} = \overline{S}_{ij} - \overline{S}^{\parallel}_{ij} \quad .$$

Carati then proposes determining the two parameters ν_1 and ν_2 by an ordinary dynamic procedure.

Model of Abba et al. Another tensor formulation was proposed by Abba *et al.* [1]. These authors propose defining the subgrid viscosity in the form of the fourth-rank tensor denoted ν_{ijkl}. This tensor is defined as the product of a scalar isotropic subgrid viscosity $\nu_{\rm iso}$ and an fourth-rank tensor denoted C, whose components are dimensionless constants which will play the role of the scalar constants ordinarily used. The tensor subgrid viscosity ν_{ijkl} thus defined is expressed:

$$\nu_{ijkl} = C_{ijkl}\nu_{\rm iso} = \left(\sum_{\alpha,\beta} C_{\alpha\beta} a_{i\alpha} a_{j\beta} a_{k\alpha} a_{l\beta}\right)\nu_{\rm iso} \quad , \tag{5.66}$$

where $a_{i\alpha}$ designates the ith component of the unit vector $\mathbf{a}_\alpha$ ($\alpha = 1,2,3$), $C_{\alpha\beta}$ is a symmetrical 3×3 matrix that replaces the scalar Smagorinsky constant. The three vectors $\mathbf{a}_\alpha$ are arbitrary and have to be defined as a function of some foreknowledge of the flow topology and its symmetries. When this information is not known, the authors propose using the local framework defined by the following three vectors:

$$\mathbf{a}_1 = \frac{\mathbf{u}}{u}, \quad \mathbf{a}_3 = \frac{\nabla(|u|^2)\times\mathbf{u}}{|\nabla(|u|^2)\times\mathbf{u}|}, \quad \mathbf{a}_2 = \mathbf{a}_3\times\mathbf{a}_1 \quad . \tag{5.67}$$

The authors apply this modification to the Smagorinsky model. The scalar viscosity is thus evaluated by the formula:

$$\nu_{\rm iso} = \overline{\Delta}^2|\overline{S}| \quad . \tag{5.68}$$

The subgrid tensor deviator is then modeled as:

$$\tau^{\rm d}_{ij} = -2\sum_{k,l} C_{ijkl}\overline{\Delta}^2|\overline{S}|\overline{S}_{kl} + \frac{2}{3}\delta_{ij}\overline{\Delta}^2 C_{mmkl}|\overline{S}|\overline{S}_{kl} \quad . \tag{5.69}$$

The model constants are then evaluated by means of a dynamic procedure.

Models Based on a Splitting Technique. Subgrid viscosity models are mostly developed in the framework of the hypotheses of the canonical analysis, *i.e.* for homogeneous turbulent flows. Experience shows that the performance of these models declines when they are used in an inhomogeneous framework, which corresponds to a non-uniform average flow. One simple idea initially proposed by Schumann [298] is to separate the velocity field into inhomogeneous and isotropic parts and to compute a specific subgrid term for each of these parts.

In practice, Schumann proposes an anisotropic subgrid viscosity model for dealing with flows whose average gradient is non-zero, and in particular any flow regions close to solid walls. The model is obtained by splitting the deviator part of the subgrid tensor $\tau^{\rm d}$ into one locally isotropic part and one inhomogeneous:

$$\tau^{\rm d}_{ij} = -2\nu_{\rm sgs}\left(\overline{S}_{ij} - \langle\overline{S}_{ij}\rangle\right) - 2\nu^{\rm a}_{\rm sgs}\langle\overline{S}_{ij}\rangle \quad , \tag{5.70}$$

where the angle brackets $\langle . \rangle$ designate an statistical average, which in practice is a spatial average in the directions of homogeneity in the solution.

The coefficients ν_{sgs} and ν^{a}_{sgs} are the scalar subgrid viscosities representing a locally isotropic turbulence and an inhomogeneous turbulence, respectively. Moin and Kim [245] and Horiuti [138] give the following definitions:

$$\nu_{sgs} = (C_1\overline{\Delta})^2 \sqrt{2\left(\overline{S}_{ij} - \langle\overline{S}_{ij}\rangle\right)\left(\overline{S}_{ij} - \langle\overline{S}_{ij}\rangle\right)} \quad , \tag{5.71}$$

$$\nu^{a}_{sgs} = (C_2\overline{\Delta}_z)^2 \sqrt{2\langle\overline{S}_{ij}\rangle\langle\overline{S}_{ij}\rangle} \quad , \tag{5.72}$$

where C_1 and C_2 are two constants. Horiuti recommends $C_1 = 0.1$ and $C_2 = 0.254$, while Moin and Kim use $C_1 = C_2 = 0.254$. The isotropic part is a function of the fluctuation of the viscosity gradients, so as to make sure that the extra-diagonal components thus predicted for the subgrid tensor cancel out on the average over time. This is consistent with the isotropic hypothesis.

The two characteristic lengths $\overline{\Delta}$ and $\overline{\Delta}_z$ represent the cutoff lengths for the two types of structures, and are evaluated as:

$$\overline{\Delta}(z) = (\overline{\Delta}_1\overline{\Delta}_2\overline{\Delta}_3)^{1/3}(1 - \exp(zu_\tau/A\nu)) \quad , \tag{5.73}$$

$$\overline{\Delta}_z(z) = \overline{\Delta}_3(1 - \exp([zu_\tau/A\nu]^2)) \quad , \tag{5.74}$$

where z is the distance to the solid wall, $\overline{\Delta}_3$ the cutoff length in the direction normal to the surface, and u_τ the friction velocity at the surface (see Sect. 9.2.1). The constant A is taken to be equal to 25.

This model was initially designed for the case of a plane channel flow. It requires being able to compute the statistical average of the velocity field, and thus can be extended only to sheared flows exhibiting at least one direction of homogeneity, or requires the use of several statistically equivalent simulations to perform the ensemble average [48, 51].

Sullivan *et al.* [321] propose a variant of it that incorporates an anisotropy factor (so that the model constant can be varied to represent the field anisotropy better):

$$\tau^{d}_{ij} = -2\nu_{sgs}\gamma\overline{S}_{ij} - 2\nu^{a}_{sgs}\langle\overline{S}_{ij}\rangle \quad . \tag{5.75}$$

The authors propose computing the viscosity ν^{a}_{sgs} as before. The ν_{sgs} term, on the other hand, is now calculated by a model with one evolution equation for the subgrid kinetic energy (see equation (4.108) in Chap. 4). Only the subgrid kinetic energy production by the isotropic is included, which is equivalent to replacing the II term in equation (4.108) with

$$2\nu_{sgs}\gamma\left(\overline{S}_{ij} - \langle\overline{S}_{ij}\rangle\right)\left(\overline{S}_{ij} - \langle\overline{S}_{ij}\rangle\right) \quad . \tag{5.76}$$

The authors evaluate the anisotropy factor from the shearing rates of the large and small scales. The average per plane of fluctuation homogeneity of the resolved strain rate tensor, calculated by

$$S' = \sqrt{2\langle\left(\overline{S}_{ij} - \langle\overline{S}_{ij}\rangle\right)\left(\overline{S}_{ij} - \langle\overline{S}_{ij}\rangle\right)\rangle} \quad , \tag{5.77}$$

is used for evaluating the shear of the small scales. The shear of the large scales is estimated as

$$S^{\diamond} = \sqrt{2\langle\overline{S}_{ij}\rangle\langle\overline{S}_{ij}\rangle} \quad . \tag{5.78}$$

The isotropy factor is evaluated as:

$$\gamma = \frac{S'}{S' + S^{\diamond}} \quad . \tag{5.79}$$